

COMPUTABILITY OF BROLIN-LYUBICH MEASURE

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ABSTRACT. Brolin-Lyubich measure λ_R of a rational endomorphism $R : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ with $\deg R \geq 2$ is the unique invariant measure of maximal entropy $h_{\lambda_R} = h_{\text{top}}(R) = \log d$. Its support is the Julia set $J(R)$. We demonstrate that λ_R is always computable by an algorithm which has access to coefficients of R , even when $J(R)$ is not computable. In the case when R is a polynomial, Brolin-Lyubich measure coincides with the harmonic measure of the basin of infinity. We find a sufficient condition for computability of the harmonic measure of a domain, which holds for the basin of infinity of a polynomial mapping, and show that computability may fail for a general domain.

1. FOREWORD

This paper continues the line of works [5, 4, 3, 6, 7] of several of the authors on algorithmic computability of Julia sets. In this brief introduction we outline our results and attempt to give a brief motivation for them.

Numerical simulation of a chaotic dynamical system: the modern paradigm. A dynamical system can be simple, and thus easy to implement numerically. Yet its orbits may exhibit a very complex behaviour. The famous paper of Lorenz [15], for example, described a rather simple nonlinear system of ordinary differential equations $\bar{x}'(t) = F(\bar{x})$ in three dimension which exhibits *chaotic* dynamics. In particular, while the flow of the system $\Phi_t(\bar{x}_0)$ is easy to calculate with an arbitrary precision for any initial value x_0 and any time t . However, any error in estimating the initial value \bar{x}_0 grows exponentially with t . This renders impractical attempting to numerically simulate the behaviour of a trajectory of the system for an extended period of time: small computational errors are magnified very rapidly. If we recall that the Lorenz system was introduced as a simple model of weather forecasting, one understands why predicting weather conditions several days in advance is difficult to do with any accuracy.

On the other hand, there is a great regularity in the global structure of a typical trajectory of Lorenz system. As was ultimately shown by Tucker [26], there exists a set $\mathcal{A} \subset \mathbb{R}^3$ such that for almost every initial point \bar{x}_0 , the limit set of the orbit,

$$\omega(\bar{x}_0) = \mathcal{A}.$$

This set is the *attractor* of the system [25, 19]. Moreover, for any continuous test function ψ , the time average of ψ along a typical orbit

$$\frac{1}{T} \int_{t=0}^T \psi(\Phi_{\bar{x}_0}(t)) dt$$

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converges to the integral $\int \psi d\mu$ with respect to a measure μ supported on \mathcal{A} .

Thus, both the spatial layout and the statistical properties of a large segment of a typical trajectory can be understood, and, indeed, simulated on a computer: even mathematicians unfamiliar with dynamics have seen the butterfly-shaped picture of Lorenz attractor \mathcal{A} . This example summarizes the modern paradigm of numerical study of chaos: while the simulation of an individual orbit for an extended period of time does not make a practical sense, one should study the limit set of a typical orbit (both as a spatial object and as a statistical distribution). A modern summary of this paradigm is found, for example, in the article of J. Palis [21].

Julia sets as counterexamples, and the topic of this paper. Julia sets are *repellers* of discrete dynamical systems generated by rational maps R of the Riemann sphere $\hat{\mathbb{C}}$ of degree $d \geq 2$. For all but finitely many points $z \in \hat{\mathbb{C}}$ the limit of the n -th preimages $R^{-n}(z)$ coincides with the Julia set $J(R)$. The dynamics of R on the set J is chaotic, again rendering numerical simulation of individual orbits impractical. Yet Julia sets are among the most drawn mathematical objects, and countless programs have been written for visualizing them.

In spite of this, two of the authors showed in [6] that there exist quadratic polynomials $f_c(z) = z^2 + c$ with the following paradoxical properties:

- an iterate $f_c(z)$ can be effectively computed with an arbitrary precision;
- there does not exist an algorithm to visualize $J(f_c)$ with an arbitrary finite precision.

This phenomenon of *non-computability* is rather subtle and rare. For a detailed exposition, the reader is referred to the monograph [7]. In practical terms it should be seen as a tale of caution in applying the above paradigm.

We cannot accurately simulate the set of limit points of the preimages $(f_c)^{-n}(z)$, but what about their statistical distribution? The question makes sense, as for all $z \neq \infty$ and every continuous test function ψ , the averages

$$\frac{1}{2^n} \sum_{w \in (f_c)^{-n}(z)} \psi(w) \xrightarrow{n \rightarrow \infty} \int \psi d\lambda,$$

where λ is the *Brolin-Lyubich probability measure* [8, 16] supported on the Julia set $J(f_c)$. We can thus ask whether the value of the integral on the right-hand side can be algorithmically computed with an arbitrary precision.

Even if $J(f_c) = \text{Supp}(\lambda)$ is not a computable set, the answer does not *a priori* have to be negative. Informally speaking, a positive answer would imply a dramatic difference between the rates of convergence in the following two limits:

$$\lim_{\text{Hausdorff}} (f_c)^{-n}(z) \longrightarrow J(f_c) \text{ and } \lim \frac{1}{2^n} \sum_{w \in (f_c)^{-n}(z)} \delta_w \xrightarrow{\text{weak}} \lambda.$$

The main results of the present paper are the following:

Theorem A. *The Brolin-Lyubich measure is always computable.*

The result of Theorem A is *uniform*, in the sense that there is a single algorithm that takes the rational map R as a parameter and computes the corresponding Brolin-Lyubich measure. Surprisingly, the proof of Theorem A does not involve much analytic machinery. The result follows from the general computable properties of the relevant space of measures.

Using the analytic tools given by the work of Drasin and Okuyama [10], or Dinh and Sibony [9], we get the following:

Theorem B. *For each rational map R , there is an algorithm $\mathcal{A}(R)$ that computes the Brolin-Lyubich measure in exponential time.*

The running time of $\mathcal{A}(R)$ will be of the form $\exp(c(R) \cdot n)$, where n is the precision parameter, and $c(R)$ is a constant that depends only on the map R (but not on n). Theorems A and B are not comparable, since Theorem B bounds the growth of the computation's running time in terms of the precision parameter, while Theorem A gives a *single* algorithm that works for all rational functions R .

Lastly, the Brolin-Lyubich measure for a polynomial coincides with the harmonic measure of the complement of the filled Julia set. As shown in [6] by two of the authors, the filled Julia set of a polynomial is always computable. In view of Theorem A, it is natural to ask what property of a computable compact set in the plane ensures computability of the harmonic measure of the complement. We show:

Theorem C. *If a closed set $K \subset \hat{\mathbb{C}}$ is computable and uniformly perfect, and has a connected complement, then the harmonic measure of the complement is computable.*

It is well-known [17] that filled Julia sets are uniformly perfect. Theorem C thus implies Theorem A in the polynomial case. Computability of the set K is not enough to ensure computability of the harmonic measure: we present a counter-example of a computable closed set with a non-computable harmonic measure of the complement.

2. JULIA SETS OF RATIONAL MAPPINGS

2.1. Dynamics on the Riemann sphere. We attempt to summarize here for convenience of a reader, unfamiliar with Complex Dynamics, the basic facts about Julia sets of rational mappings. An excellent book of Milnor [20] presents a detailed and self-contained introduction to the subject; proofs of most of the facts we state can be found there.

We first recall that the Riemann sphere $\hat{\mathbb{C}}$ is the Riemann surface with the topological type of the 2-sphere, S^2 . Such a complex-analytic manifold can be constructed by gluing together two copies of the complex plane $C_1 = \mathbb{C}$, $C_2 = \mathbb{C}$ by identifying $z \in C_1 \setminus \{0\}$ with $w = 1/z \in C_2$. This procedure can be loosely described as adjoining a point at infinity to the complex plane C_1 – we will denote ∞ the origin in C_2 (so that “ $\infty = 1/0$ ”). It is convenient sometimes to visualize $\hat{\mathbb{C}}$ as the unit sphere

$$S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3.$$

To this end, consider the stereographic projection from the “north pole” $(0, 0, 1) \in S^2$, which sends $S^2 \setminus \{(0, 0, 1)\}$ to the plane $z = 0$ which we naturally identify with $C_1 = \mathbb{C}$ by $z = x + iy$. In this model, the north pole becomes the point at infinity.

The Euclidean metric on \mathbb{R}^3 restricted to S^2 is transferred by the stereographic projection to the *spherical metric* on \mathbb{C} . This metric is given by

$$ds^2 = \left(\frac{2}{1 + |z|^2} \right) |dz|^2.$$

We will refer to the spherical distance as $d_{\hat{\mathbb{C}}}$, as opposed to the usual Euclidean distance d .

A rational function $R(z) = P(z)/Q(z)$ induces an analytic covering $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ branched at the finitely many *critical points* $\zeta \in \hat{\mathbb{C}}$ with $R'(\zeta) = 0$. The degree d of this covering is finite, and coincides with the algebraic degree of R :

$$d = \max(\deg(P), \deg(Q)),$$

assuming P and Q have no common factors. Every analytic branched covering of $\hat{\mathbb{C}}$ of a finite degree is given by a rational function.

We will consider a rational mapping R of degree $\deg R = d \geq 2$ (that is, non-linear) as a dynamical system on the Riemann sphere; and denote R^n the n -th iterate of R . The R -orbit of a point ζ is the sequence $\{R^n(\zeta)\}_{n=0}^\infty$. The Julia set is defined as the complement of the set where the dynamics is Lyapunov-stable:

Definition 2.1. Denote $F(R)$ the set of points $z \in \hat{\mathbb{C}}$ having an open neighborhood $U(z)$ on which the family of iterates $R^n|_{U(z)}$ is equicontinuous; that is for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d_{\hat{\mathbb{C}}}(z, w) < \delta$ then for every $n \in \mathbb{N}$ one has $d_{\hat{\mathbb{C}}}(R^n(z), R^n(w)) < \epsilon$. The set $F(R)$ is called the Fatou set of R and its complement $J(R) = \hat{\mathbb{C}} \setminus F(R)$ is the Julia set.

In the case when the rational mapping is a polynomial

$$P(z) = a_0 + a_1z + \cdots + a_dz^d : \mathbb{C} \rightarrow \mathbb{C}$$

an equivalent way of defining the Julia set is as follows. Obviously, there exists a neighborhood of ∞ on $\hat{\mathbb{C}}$ on which the iterates of P uniformly converge to ∞ . Denoting $A(\infty)$ the maximal such domain of attraction of ∞ we have $A(\infty) \subset F(R)$. We then have

$$J(P) = \partial A(\infty).$$

The bounded set $\hat{\mathbb{C}} \setminus A(\infty)$ is called *the filled Julia set*, and denoted $K(P)$; it consists of points whose orbits under P remain bounded:

$$K(P) = \{z \in \hat{\mathbb{C}} \mid \sup_n |P^n(z)| < \infty\}.$$

For future reference, let us summarize in a proposition below the main properties of Julia sets:

Proposition 2.1. *Let $R : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational function. Then the following properties hold:*

- (a) $J(R)$ is a non-empty compact subset of $\hat{\mathbb{C}}$ which is completely invariant: $R^{-1}(J(R)) = J(R)$;
- (b) $J(R) = J(R^n)$ for all $n \in \mathbb{N}$;
- (c) $J(R)$ has no isolated points;
- (d) if $J(R)$ has non-empty interior, then it is the whole of $\hat{\mathbb{C}}$;
- (e) let $U \subset \hat{\mathbb{C}}$ be any open set with $U \cap J(R) \neq \emptyset$. Then there exists $n \in \mathbb{N}$ such that $R^n(U) \supset J(R)$;
- (f) periodic orbits of R are dense in $J(R)$.

Let us further comment on the last property. For a periodic point $z_0 = R^p(z_0)$ of period p its *multiplier* is the quantity $\lambda = \lambda(z_0) = DR^p(z_0)$. We may speak of the multiplier of a periodic cycle, as it is the same for all points in the cycle by the Chain Rule. In the case when $|\lambda| \neq 1$, the dynamics in a sufficiently small neighborhood of the cycle is governed by the Mean Value Theorem: when $|\lambda| < 1$, the cycle is

attracting (*super-attracting* if $\lambda = 0$), if $|\lambda| > 1$ it is *repelling*. All repelling periodic points are in the Julia set, and all attracting ones are in the Fatou set.

The situation is much more complicated when $|\lambda| = 1$; understanding of the local dynamics in this case is not yet complete.

One of the founders of the subject, P. Fatou, has shown that for a rational mapping R with $\deg R = d \geq 2$ at most finitely many periodic orbits are non-repelling. A sharp bound on their number depending on d has been established by Shishikura; it is equal to the number of critical points of R counted with multiplicity:

Fatou-Shishikura Bound. For a rational mapping of degree d the number of the non-repelling periodic cycles taken together with the number of cycles of Herman rings is at most $2d - 2$. For a polynomial of degree d the number of non-repelling periodic cycles in \mathbb{C} is at most $d - 1$.

Therefore, we may refine the last statement of Proposition 2.1:

$$(f') \quad J(R) = \overline{\{\text{repelling periodic orbits of } R\}}.$$

We also note a useful corollary of Proposition 2.1 (e):

Corollary 2.2. *Let $w \in J(R)$. Then*

$$J(R) = \overline{\bigcup_{k \geq 0} f^{-k}(w)}.$$

To conclude the discussion of the basic properties of Julia sets, let us consider the simplest examples of non-linear rational endomorphisms of the Riemann sphere, the quadratic polynomials. Every affine conjugacy class of quadratic polynomials has a unique representative of the form $f_c(z) = z^2 + c$, the family

$$f_c(z) = z^2 + c, \quad c \in \mathbb{C}$$

is often referred to as *the quadratic family*. For a quadratic map the structure of the Julia set is governed by the behavior of the orbit of the only finite critical point 0. In particular, the following dichotomy holds:

Proposition 2.3. *Let $K = K(f_c)$ denote the filled Julia set of f_c , and $J = J(f_c) = \partial K$. Then:*

- $0 \in K$ implies that K is a connected, compact subset of the plane with connected complement;
- $0 \notin K$ implies that $K = J$ is a planar Cantor set.

The *Mandelbrot set* $\mathcal{M} \subset \mathbb{C}$ is defined as the set of parameter values c for which $J(f_c)$ is connected.

2.2. Brolin-Lyubich measure on the Julia set.

Definition 2.2. Consider a rational map $R : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ of degree $d \geq 2$. We say that a probability measure μ on $\hat{\mathbb{C}}$ is *balanced* (with respect to R) if for every set $X \subset \hat{\mathbb{C}}$ on which R is injective we have

$$\mu(R(X)) = d \cdot \mu(X),$$

that is, the Jacobian of μ is equal to d .

We see that a balanced measure μ is necessarily invariant: as most points in $\hat{\mathbb{C}}$ have d preimages under R ,

$$\mu(R^{-1}(X)) = \mu(X).$$

However, a rational map has many invariant probability measures (as a simplistic example, for a periodic orbit $z_0 \mapsto z_1 \mapsto \cdots \mapsto z_{p-1} \mapsto z_0$ define $\mu = \frac{1}{p} \sum \delta_{z_i}$). On the other hand there is exactly one balanced measure for R : the Brolin-Lyubich measure λ . Constructed by Brolin [8] for polynomials, and later by Lyubich [16] for a general rational function, it is supported on the Julia set $J(R)$. Lyubich showed that for all but finitely many points $z \in \hat{\mathbb{C}}$ the weak limit

$$\lim_{n \rightarrow \infty} \frac{1}{d^n} \sum_{w \in R^{-n}(z)} \delta_w = \lambda. \quad (2.1)$$

In general, given a transformation T of a compact space X , denote by $h_{\text{top}}(T)$ and $h_\mu(T)$ the topological and measure-theoretic entropies, respectively. The well known *Variational Principle*, tell us that:

$$h_{\text{top}}(T) = \sup_{\mu \in \mathcal{M}_T} h_\mu(T),$$

where \mathcal{M}_T denotes the set of T -invariant measures. A measure μ is called a *measure of maximal entropy* if $h_\mu(T) = h_{\text{top}}(T)$.

Lyubich showed that λ is the unique *measure of maximal entropy* of R :

Theorem 2.4 ([16]). *The measure λ is the unique measure on $\hat{\mathbb{C}}$ for which the metric entropy $h_\lambda(R)$ coincides with the topological entropy of R :*

$$h_\lambda(R) = h_{\text{top}}(R) = \log d.$$

Note that for any invariant measure μ we have

$$\int \text{Jac}_\mu f d\mu \leq d,$$

therefore a measure of maximal entropy is necessarily balanced.

2.3. Harmonic measure in polynomial dynamics. A detailed discussion of harmonic measure can be found in [12]. Here we briefly recall some of the relevant facts.

Let G be a simply-connected domain in $\hat{\mathbb{C}}$ whose complement K contains at least two points, and $g \in G$. The *harmonic measure* $\omega_{G,g}$ is defined on the boundary ∂G . For a set $E \subset \partial G$ it is equal to the probability that a Brownian path originating at g will first hit ∂G within the set E .

To define the harmonic measure for a non simply-connected domain $G \equiv \hat{\mathbb{C}} \setminus K$ we have to require that a Brownian path originating in G will hit ∂G almost surely, a condition which is satisfied automatically for a simply-connected domain. A quantitative measure of a likelihood that such a set will be hit by a Brownian path is defined as follows. Consider $K \Subset \mathbb{C}$, and let B_t be a Brownian path which is started uniformly at a circle $\{|z| = R\}$ which surrounds K . Denote τ the first moment when $B_\tau \in K$. The *logarithmic capacity* of K is

$$\text{Cap}(K) = \exp(\mathbb{E}(\log |B_\tau|)).$$

By way of an example, consider a connected and locally-connected compact set $K \subset \mathbb{C}$. In this case, ∂G is a continuous image of the unit circle. In fact, consider the unique conformal Riemann mapping

$$\psi : G \equiv \hat{\mathbb{C}} \setminus K \rightarrow \hat{\mathbb{C}} \setminus \overline{D_R(0)} \equiv (D_R(0))^c, \text{ with } \psi(\infty) = \infty \text{ and } \psi'(\infty) = 1.$$

The quantity $r(G, \infty) \equiv 1/R$ is the *conformal radius* of G about ∞ .

By a classical theorem of Carathéodory, ψ^{-1} extends continuously to map $\bar{G} \rightarrow \hat{\mathbb{C}} \setminus D_R(0)$. By symmetry considerations, the harmonic measure $\omega_{(D_R(0))^c, \infty}$ coincides with the Lebesgue measure μ on the circle $\partial D_R(0) = \{|z| = R\}$. Conformal invariance of Brownian motion implies that $\omega_{G, \infty}$ is obtained by pushing forward μ by $\psi^{-1}|_{\partial D_R(0)}$, and that

$$\text{Cap}(G) = 1/r(G, \infty).$$

Consider a polynomial $P : \mathbb{C} \rightarrow \mathbb{C}$ with $\deg P \geq 2$. The capacity of the filled Julia set $K(P)$ is equal to one. This follows from a classical result of Böttcher when $K(P)$ is connected (see [8] for the general case). Brolin [8] was the first to show that the balanced measure λ of P coincides with the harmonic measure $\omega_{\hat{\mathbb{C}} \setminus K(P), \infty}$.

3. COMPUTABILITY

3.1. Algorithms and computable functions on integers. The notion of an algorithm was formalized in the 30's, independently by Post, Markov, Church, and, most famously, Turing. Each of them proposed a model of computation which determines a set of integer functions that can be *computed* by some mechanical or algorithmic procedure. Later on, all these models were shown to be equivalent, so that they define the same class of integer functions, which are now called *computable (or recursive) functions*. It is standard in Computer Science to formalize an algorithm as a *Turing Machine* [27]. We will not define it here, and instead will refer an interested reader to any standard introductory textbook in the subject. It is more intuitively familiar, and provably equivalent, to think of an algorithm as a program written in any standard programming language.

In any programming language there is only a countable number of possible algorithms. Fixing the language, we can enumerate them all (for instance, lexicographically). Given such an ordered list $(\mathcal{A}_n)_{n=1}^\infty$ of all algorithms, the index n is usually called the *Gödel number* of the algorithm \mathcal{A}_n .

We will call a function $f : \mathbb{N} \rightarrow \mathbb{N}$ *computable* (or *recursive*), if there exists an algorithm \mathcal{A} which, upon input n , outputs $f(n)$. Computable functions of several integer variables are defined in the same way.

A function $f : W \rightarrow \mathbb{N}$, which is defined on a subset $W \subset \mathbb{N}$, is called *partial recursive* if there exists an algorithm \mathcal{A} which outputs $f(n)$ on input $n \in W$, and runs forever if the input $n \notin W$.

3.2. Time complexity of a problem. For an algorithm \mathcal{A} with input w the *running time* is the number of steps \mathcal{A} makes before terminating with an output. The *size* of an input w is the number of dyadic bits required to specify w . Thus for $w \in \mathbb{N}$, the size of w is the integer part of $\log_2 w$. The *running time of \mathcal{A}* is the function

$$T_{\mathcal{A}} : \mathbb{N} \rightarrow \mathbb{N}$$

such that

$$T_{\mathcal{A}}(n) = \max\{\text{the running time of } \mathcal{A}(w) \text{ for inputs } w \text{ of size } n\}.$$

In other words, $T_{\mathcal{A}}(n)$ is the worst case running time for inputs of size n . For a computable function $f : \mathbb{N} \rightarrow \{0, 1\}$ the time complexity of f is said to have an upper bound $T(n)$ if there exists an algorithm \mathcal{A} with running time bounded by $T(n)$ that computes f .

3.3. Computable and semi-computable sets of natural numbers. A set $E \subseteq \mathbb{N}$ is said to be *computable* if its characteristic function $\chi_E : \mathbb{N} \rightarrow \{0, 1\}$ is computable. That is, if there is an algorithm $\mathcal{A} : \mathbb{N} \rightarrow \{0, 1\}$ that, upon input n , halts and outputs 1 if $n \in E$ or 0 if $n \notin E$. Such an algorithm allows to *decide* whether or not a number n is an element of E . Computable sets are also called *recursive* or *decidable*.

Since there are only countably many algorithms, there exist only countably many computable subsets of \mathbb{N} . A well known “explicit” example of a non computable set is given by the *Halting set*

$$H := \{i \text{ such that } \mathcal{A}_i \text{ halts}\}.$$

Turing [27] has shown that there is no algorithmic procedure to decide, for any $i \in \mathbb{N}$, whether or not the algorithm with Gödel number i , \mathcal{A}_i , will eventually halt.

On the other hand, it is easy to describe an algorithmic procedure which, on input i , will halt if $i \in H$, and will run forever if $i \notin H$. Such a procedure can informally be described as follows: *on input i emulate the algorithm \mathcal{A}_i ; if \mathcal{A}_i halts then halt.*

In general, we will say that a set $E \subset \mathbb{N}$ is *lower-computable* (or *semi-decidable*, or *recursively enumerable*) if there exists an algorithm \mathcal{A}_E which on an input n halts if $n \in E$, and never halts otherwise. Thus, the algorithm \mathcal{A}_E can verify the inclusion $n \in E$, but not the inclusion $n \in E^c$. We say that \mathcal{A}_E *semi-decides* $n \in E$ (or semi-decides E). The complement of a lower-computable set is called *upper-computable*.

The following is an easy exercise:

Proposition 3.1. *A set is computable if and only if it is simultaneously upper- and lower-computable.*

3.4. Computability over the reals. Strictly speaking, algorithms only work on natural numbers, but this can be easily extended to the objects of any countable set once a bijection with integers has been established. The operative power of an algorithm on the objects of such a numbered set obviously depends on what can be algorithmically recovered from their numbers. For example, the set \mathbb{Q} of rational numbers can be injectively numbered $\mathbb{Q} = \{q_0, q_1, \dots\}$ in an *effective* way: the number i of a rational a/b can be computed from a and b , and vice versa. The abilities of algorithms on integers are then transferred to the rationals. For instance, algorithms can perform algebraic operations and decide whether or not $q_i > q_j$ (in the sense that the set $\{(i, j) : q_i > q_j\}$ is decidable).

Extending algorithmic notions to functions of real numbers was pioneered by Banach and Mazur [1, 18], and is now known under the name of *Computable Analysis*. Let us begin by giving the definition of a computable real number, going back to the seminal paper of Turing [27].

Definition 3.1. A real number x is called

- *computable* if there is a computable function $f : \mathbb{N} \rightarrow \mathbb{Q}$ such that

$$|f(n) - x| < 2^{-n};$$

- *lower-computable* if there is a computable function $f : \mathbb{N} \rightarrow \mathbb{Q}$ such that

$$f(n) \nearrow x;$$

- *upper-computable* if there is a computable function $f : \mathbb{N} \rightarrow \mathbb{Q}$ such that

$$f(n) \searrow x.$$

Algebraic numbers or the familiar constants such as π , e , or the Feigenbaum constant are all computable. However, the set of all computable numbers \mathbb{R}_C is necessarily countable, as there are only countably many computable functions.

We also remark that if x is lower-computable then there is an algorithm to semi-decide the set $\{q_i < x\}$: just compute $f(n)$ for each n and halt if $q_i < f(n)$. In other words, the set $\{q \in \mathbb{Q} : q < x\}$ is lower-computable. The converse is also obviously true:

Proposition 3.2. *If $E \subset \mathbb{Q}$ is lower-computable and $x = \sup E < \infty$, then x is lower-computable.*

In the same way as there exist lower-computable sets which are not computable, there exists lower-computable numbers which are not computable. The usual construction is as follows: let $(a_i)_i$ be an algorithmic enumeration (without repetitions) of a lower-computable set A which is not computable. For instance, we can take

$$A = \{i \in \mathbb{N} \text{ such that } \mathcal{A}_i \text{ halts}\}.$$

Define

$$q_n = \sum_{i=0}^n 2^{-a_i-1}.$$

Clearly, $(q_n)_n$ is a computable non-decreasing sequence of rational numbers. Being bounded by 1, it converges. The limit, say x , is then a lower-computable number. If x were computable, it would be possible to compute the binary expansion of x which, in turn, would allow to decide the set A .

We also note:

Proposition 3.3. *A real number x is computable if and only if it is simultaneously lower- and upper-computable.*

Proof. Let us assume that x is both lower- and upper-computable. Thus there exist algorithms \mathcal{A}_1 and \mathcal{A}_2 which compute sequences of rationals a_j and b_j respectively with

$$a_j \nearrow x \text{ and } b_j \searrow x.$$

Consider the algorithm \mathcal{A} which on the input n emulates \mathcal{A}_1 , \mathcal{A}_2 to find the first $k(n)$ such that $|a_{k(n)} - b_{k(n)}| < 2^{-n}$, and then outputs $a_{k(n)}$. Then $f(n) = a_{k(n)}$ is a computable function such that $|f(n) - x| < 2^{-n}$ and hence $x \in \mathbb{R}_C$.

The other direction is trivial. \square

3.5. Uniform computability. In this paper we will use algorithms to define *computability* notions on more general objects. Depending on the context, these objects will take particular names (computable, lower-computable, etc...) but the definition will always follow the scheme:

an object x is computable if there exists an algorithm \mathcal{A} satisfying the property $P(\mathcal{A}, x)$.

For example, a real number x is *computable* if there exists an algorithm \mathcal{A} which computes a function $f : \mathbb{N} \rightarrow \mathbb{Q}$ satisfying $|f(n) - x| < 2^{-n}$ for all n . Each time such definition is made, a *uniform version* will be implicitly defined:

the objects $\{x_\gamma\}_{\gamma \in \Gamma}$ are computable uniformly on a countable set Γ if there exists an algorithm \mathcal{A} with an input $\gamma \in \Gamma$, such that for all $\gamma \in \Gamma$, $\mathcal{A}_\gamma := \mathcal{A}(\gamma, \cdot)$ satisfies the property $P(\mathcal{A}_\gamma, x_\gamma)$.

In our example, a sequence of reals $(x_i)_i$ is *computable uniformly in i* if there exists \mathcal{A} with two natural inputs i and n which computes a function $f(i, n) : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}$ such that for all $i \in \mathbb{N}$, the values of the function $f_i(\cdot) := f(i, \cdot)$ satisfy

$$|f_i(n) - x_i| < 2^{-n} \text{ for all } n \in \mathbb{N}.$$

3.6. Computable metric spaces. The above definitions equip the real numbers with a computability structure. This can be extended to virtually any separable metric space, making them *computable metric spaces*. We now give a short introduction. For more details, see [28].

Definition 3.2. A *computable metric space* is a triple (X, d, \mathcal{S}) where:

- (1) (X, d) is a separable metric space,
- (2) $\mathcal{S} = \{s_i : i \in \mathbb{N}\}$ is a dense sequence of points in X ,
- (3) $d(s_i, s_j)$ are computable real numbers, uniformly in (i, j) .

The points in \mathcal{S} are called *ideal*.

Example 3.1. A basic example is to take the space $X = \mathbb{R}^n$ with the usual notion of Euclidean distance $d(\cdot, \cdot)$, and to let the set \mathcal{S} consist of points $\bar{x} = (x_1, \dots, x_n)$ with rational coordinates. In what follows, we will implicitly make these choices of \mathcal{S} and $d(\cdot, \cdot)$ when discussing computability in \mathbb{R}^n .

Definition 3.3. A point x is *computable* if there is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$|s_{f(n)} - x| < 2^{-n} \text{ for all } n.$$

If $x \in X$ and $r > 0$, the metric ball $B(x, r)$ is defined as

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

Since the set $\mathcal{B} := \{B(s, q) : s \in \mathcal{S}, q \in \mathbb{Q}, q > 0\}$ of *ideal balls* is countable, we can fix an enumeration $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$.

Proposition 3.4. A point x is computable if and only if the relation $x \in B_i$ is semi-decidable, uniformly in i .

Proof. Assume first that x is computable. We have to show that there is an algorithm \mathcal{A} which inputs a natural number i and halts if and only if $x \in B_i$. Since x is computable, for any n we can produce an ideal point s_n satisfying $|s_n - x| < 2^{-n}$. The algorithm \mathcal{A} work as follows: upon input i , it computes the center and radius of B_i , say s and r . It then searches for $n \in \mathbb{N}$ such that

$$d(s_n, s) + 2^{-n} < r.$$

Evidently, the above inequality will hold for some n if and only if $x \in B_i$.

Conversely, assume that the relation $x \in B_i$ is semi-decidable uniformly in i . To produce an ideal point s_n satisfying $|s_n - x| < 2^{-n}$, we only need to enumerate all

ideal balls of radius 2^{-n+1} until one containing x is found. We can take s_n to be the center of this ball. \square

Definition 3.4. An open set U is called *lower-computable* if there is a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$U = \bigcup_{n \in \mathbb{N}} B_{f(n)}.$$

Example 3.2. Let $\epsilon > 0$ be a lower-computable real. Then the ball $B(0, \epsilon)$ is a lower-computable open set. Indeed: $B(s_0, \epsilon) = \bigcup_n B(0, q_n)$, where $(q_n)_n$ is the computable sequence converging to ϵ from below.

It is not difficult to see that finite intersections or infinite unions of (uniformly) lower-computable open sets are again lower computable. As in Proposition (3.4), one can show that the relation $x \in U$ is semi-decidable for a computable point x and an open lower-computable set.

We will now introduce computable functions. Let X' be another computable metric space with idea balls $\mathcal{B}' = \{B'_i\}$.

Definition 3.5. A function $f : X \rightarrow X'$ is *computable* if the sets $f^{-1}(B'_i)$ are lower-computable open, uniformly in i .

An immediate corollary of the definition is:

Proposition 3.5. *Every computable function is continuous.*

The above definition of a computable function is concise, yet not very transparent. To give its $\epsilon - \delta$ version, we need another concept. We say that a function $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is an *oracle* for $x \in X$ if

$$d(s_{\phi(m)}, x) < 2^{-m}.$$

An algorithm may *query* an oracle by reading the values of the function ϕ for an arbitrary $n \in \mathbb{N}$. We have the following:

Proposition 3.6. *A function $f : X \rightarrow X'$ is computable if and only if there exists an algorithm \mathcal{A} with an oracle for $x \in X$ and an input $n \in \mathbb{N}$ which outputs $s'_n \in \mathcal{S}'$ such that $d(s'_n, f(x)) < 2^{-n}$.*

In other words, given an arbitrarily good approximation of the input of f it is possible to constructively approximate the value of f with any desired precision.

3.6.1. *Computability of closed sets.* Having dfined lower-computable open sets, we naturally proceed to the following definition.

Definition 3.6. A closed set K is *upper-computable* if its complement is lower-computable.

Let us look at two examples. Firstly,

Example 3.3. A closed ideal ball $\text{cl}(B(s, q))$ is clearly upper-computable. To see this, observe that a point s_n belongs to $X \setminus \text{cl}(B(s, q))$ if and only if $d(s_n, s) > q$. Since this last relation is semi-decidable, we can enumerate such ideal points. Moreover, for each of them we can also find q_n satisfying $0 < q_n < d(s_n, s) - q$, so that $B(s_n, q_n) \subset X \setminus \text{cl}(B(s, q))$.

Our second example is more interesting:

Example 3.4. Let $P : \mathbb{C} \rightarrow \mathbb{C}$ be a computable polynomial of degree $\deg P \geq 2$. Then the filled Julia set $K(P)$ is upper-computable.

Proof. Indeed, let $M \in \mathbb{Q}$ be such that $K(P) \subset B(0, M)$. Enumerate the points in \mathbb{R}^2 with rational coordinates $\mathcal{S} = \{s_n = (a_n, b_n)\}$, and set $\zeta_n = a_n + ib_n$. For every point $\zeta_n \in \mathbb{C} \setminus K(P)$ we can identify an iterate $P^l(\zeta_n) \notin B(0, M)$. Moreover, for such a point we can find $\epsilon_n \in \mathbb{Q}$ such that

$$P^l(B(\zeta_n, \epsilon_n)) \cap B(0, M) = \emptyset.$$

We can thus algorithmically enumerate a sequence of open ideal balls which exhausts $\mathbb{R}^2 \setminus K(P)$. \square

Definition 3.7. A closed set K is *lower-computable* if the relation $K \cap B_i \neq \emptyset$ is semi-decidable, uniformly in i .

In other words, a closed set K is lower-computable if there exists an algorithm \mathcal{A} which enumerates all ideal balls which have non-empty intersection with K .

To see that this definition is a natural extension of lower computability of open sets, we note:

Example 3.5.

- (1) The closure of an ideal ball $\text{cl}(B(s, q))$ is lower-computable. Indeed, $B(s_i, q_i) \cap \text{cl}(B(s, q)) \neq \emptyset$ if and only if $d(s, s_n) < q + q_n$.
- (2) More generally, the closure $\text{cl}(U)$ of any open lower-computable set U is lower-computable since $B_i \cap \text{cl}(U) \neq \emptyset$ if and only if there exists $s \in B_i \cap U$.

The following is a useful characterization of lower-computable sets:

Proposition 3.7. A closed set K is lower-computable if and only if there exists a sequence of uniformly computable points $x_i \in K$ which is dense in K .

Proof. Observe that, given some ideal ball $B = B(s, q)$ intersecting K , the relations $\text{cl}(B_i) \subset B$, $q_i \leq 2^{-k}$ and $B_i \cap K \neq \emptyset$ are all semi-decidable and then we can find an exponentially decreasing sequence of ideal balls (B_k) intersecting K . Hence $\{x\} = \bigcap_k B_k$ is a computable point lying in $B \cap K$.

The other direction is obvious. \square

Example 3.6. Let R be a computable rational map of degree $\deg R \geq 2$. Then the Julia set $J(R)$ is lower-computable.

Sketch of proof. We will use Corollary 2.2. Periodic points of R are computable (by any standard root-finding algorithm) and so are their multipliers. We can semi-decide whether a periodic point is repelling (if the multiplier is greater than 1 we will be able to establish this with a certainty by computing the point and its multiplier precisely enough). Therefore, the repelling periodic points of R are computable. Let w be any such point. The points in $\bigcup_{k \geq 0} f^{-k}(w)$ are uniformly computable, and dense in $J(R)$. By Proposition 3.7, $J(R)$ is a closed lower-computable set. \square

Definition 3.8. A closed set is *computable* if it is lower and upper computable.

Putting together Examples 3.4 and 3.6, we obtain the following theorem of [4]:

Example 3.7. Let P be a computable polynomial with $\deg P \geq 2$, and suppose that $K(P)$ has empty interior, that is, $K(P) = J(P)$. Then $K(P)$ is a computable set.

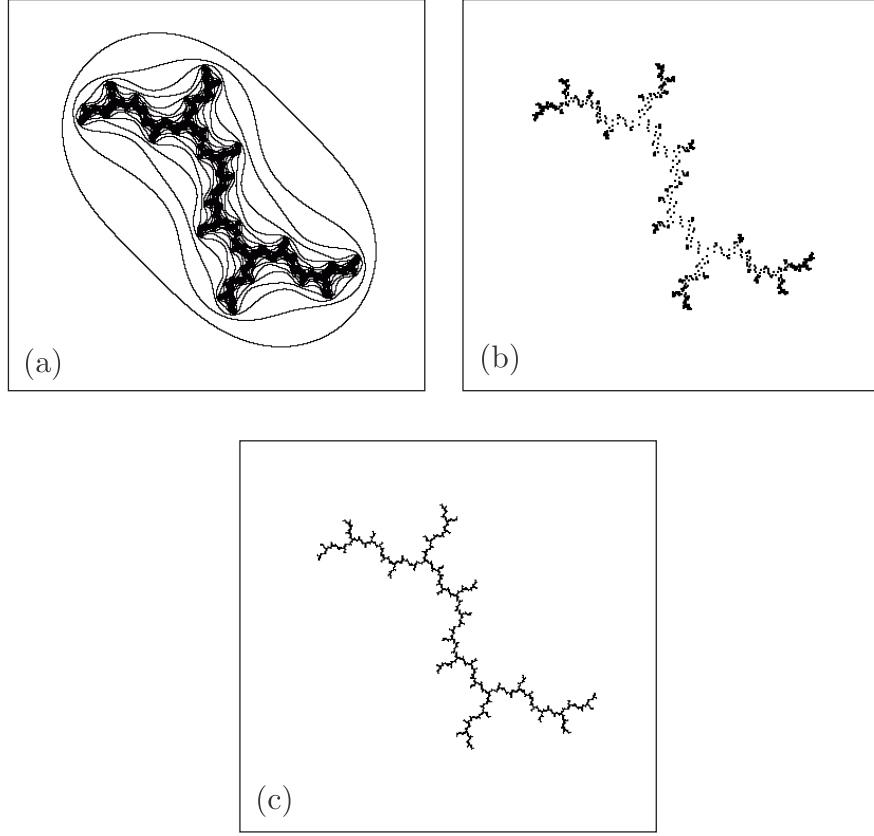


FIGURE 1. Computing the Julia set of $f(z) = z^2 + i$ ($J(f) = K(f)$). (a) approximating the filled Julia set from above: the first 15 preimages of a large disk $D = B(0, R) \supset K(f)$; (b) approximating the Julia set from below: $\cup_{0 \leq k \leq 12} f^{-k}(\beta)$ where β is a repelling fixed point in $J(f)$; (c) a good-quality picture of $J(f)$.

Here is an alternative way to define a computable set. Recall that *Hausdorff distance* between two compact sets K_1, K_2 is

$$\text{dist}_H(K_1, K_2) = \inf_{\epsilon} \{K_1 \subset U_{\epsilon}(K_2) \text{ and } K_2 \subset U_{\epsilon}(K_1)\},$$

where $U_{\epsilon}(K) = \bigcup_{z \in K} B(z, \epsilon)$ stands for an ϵ -neighborhood of a set. The set of all compact subsets of M equipped with Hausdorff distance is a metric space which we will denote by $\text{Comp}(M)$. If M is a computable metric space, then $\text{Comp}(M)$ inherits this property; the ideal points in $\text{Comp}(M)$ are finite unions of closed ideal balls in M . We then have the following:

Proposition 3.8. *A set $K \in M$ is computable if and only if it is a computable point in $\text{Comp}(M)$.*

Equivalently, K is computable if there exists an algorithm \mathcal{A} with a single natural input n , which outputs a finite collection of closed ideal balls $\text{cl } B_1, \dots, \text{cl } B_{i_n}$ such

that

$$\text{dist}_H(\bigcup \text{cl } B_{i_n}, K) < 2^{-n}.$$

3.7. Computable probability measures. Let $\mathcal{M}(X)$ denote the set of Borel probability measures over a metric space X . We recall the notion of weak convergence of measures:

Definition 3.9. A sequence of measures $\mu_n \in M(X)$ is said to be *weakly convergent* to $\mu \in M(X)$ if $\int f d\mu_n \rightarrow \int f d\mu$ for each $f \in C_0(X)$.

Any smaller family of functions characterizing the weak convergence is called *sufficient*. It is well-known, that when X is a compact separable and complete metric space, then so is $\mathcal{M}(X)$.

Weak convergence on $M(X)$ is compatible with the notion of *Wasserstein-Kantorovich distance*, defined by:

$$W_1(\mu, \nu) = \sup_{f \in 1\text{-Lip}(X)} \left| \int f d\mu - \int f d\nu \right|$$

where $1\text{-Lip}(X)$ is the space of Lipschitz functions on X , having Lipschitz constant less than one.

The following result (see [13]) says that, when X is a computable metric space, $\mathcal{M}(X)$ inherits its computability structure.

Proposition 3.9. *Let \mathcal{D} be the set of finite convex rational combinations of Dirac measures supported on ideal points of X . Then the triple $(\mathcal{M}(X), W_1, \mathcal{D})$ is a computable metric space.*

Definition 3.1. A *computable measure* is a computable point in $(\mathcal{M}(X), W_1, \mathcal{D})$. That is, it is a measure which can be algorithmically approximated in the weak sense by *discrete* measures with any given precision.

As examples of computable measures, consider the Lebesgue measure in \mathbb{R}^n , or any smooth measure in \mathbb{R}^n with a computable density function.

The following proposition (see [13]) gives a useful characterization of the computability of the measure.

Proposition 3.10. *Let μ be a Borel probability measure. The following statements are equivalent:*

- (1) μ is computable,
- (2) $\mu(B_{i_1} \cup \dots \cup B_{i_n})$ are lower-computable, uniformly in i_1, \dots, i_n ,
- (3) for any uniformly computable sequence of functions $(f_i)_i$, the integral $\int f_i d\mu$ is computable uniformly in i .

We will also need the following fact (see [24]), :

Proposition 3.11. *If $(f_i)_i$ is a uniformly computable sequence of functions, then the integral operators*

$$L_i : \mathcal{M}(X) \rightarrow \mathbb{R} \text{ defined by } L_i(\mu) := \int f_i d\mu,$$

are uniformly computable.

To illustrate the concepts we have introduced, we end this section by constructing an example of a computable set $E \subset I = [0, 1]$ such that Lebesgue measure restricted to it is not computable. Indeed, any non-atomic probability measure assigning positive measure to intervals in E , will not be computable.

Example 3.8. Let \mathcal{A}_i be a Gödel ordering of algorithms. Set $a_0 = 0$ and $a_i = a_{i-1} + 2^{-i}$. Define the set S_i to be:

$$S_i = \begin{cases} (a_i, a_{i+1}) \setminus \{a_i + n2^{-j-i} : n = 1, \dots, 2^j - 1\}, & \mathcal{A}_i \text{ halts in } j \text{ steps} \\ \emptyset, & \mathcal{A}_i \text{ does not halt.} \end{cases}$$

Our set is defined by

$$E = I \setminus \cup_i S_i.$$

Clearly, $\cup_i S_i$ is lower-computable open and thus E is upper computable.

Let us prove that E is also lower-computable by producing a dense computable sequence of points in E . To this end, we run an algorithm which at step j simulates all algorithms \mathcal{A}_i , $i \leq j$ for the first j steps (or until they halt). For every i such that \mathcal{A}_i does not halt in fewer than j steps it then outputs the set

$$\{a_i + n2^{-j-i} : n = 1, \dots, 2^j - 1\}.$$

We denote E_j the union of the sets output by the algorithm at step j .

It is clear that

$$E = \text{cl}(\cup_j E_j).$$

Thus E is lower-computable, and hence, computable.

Suppose μ is a non atomic probability measure on E assigning positive mass to every interval in E . Then, for each i , $\mu(a_i, a_{i+1}) > 0$ if and only if \mathcal{A}_i does not halt.

Let us assume that μ is a computable measure on E . Then, by Theorem 3.10, the relation $\mu(I) > 0$ is semi-decidable for any rational interval I . Hence the Halting set is upper-computable. Since it is also lower-computable, the Halting set is computable. We have thus arrived at a contradiction with the undecidability of the Halting problem.

4. COMPUTABILITY OF BROLIN-LYUBICH MEASURE

4.1. Some preliminaries. In what follows we will require the following facts. The first theorem is classical, see e.g. [20].

Koebe 1/4 Theorem. *If f is a univalent function on a disk $D \equiv B(z_0, r) \subset \mathbb{C}$, then*

$$\text{dist}(f(z_0), \partial(f(D))) \geq \frac{1}{4} |f'(z_0)| r$$

Considerations of compactness (see [11]) imply that there is an algorithmic procedure to semi-decide whether a given lower-computable open set of probability measures on $\hat{\mathbb{C}}$ contains the whole $\mathcal{M}(\hat{\mathbb{C}})$. It will be convenient for us to use a uniform version of this statement:

Proposition 4.1. *Let $(U_i)_i$ be a sequence of uniformly lower-computable open subsets of $\mathcal{M}(\hat{\mathbb{C}})$. Then the relation $\mathcal{M}(\hat{\mathbb{C}}) \subseteq U_i$ is semi-decidable, uniformly in i .*

Sketch of proof. It is enough that for any given finite list of ideal balls $\{B_{k_i}\}_{i=1}^m$, we can semi-decide the relation

$$\mathcal{M}(\hat{\mathbb{C}}) \subseteq \bigcup_{i \leq m} B_{k_i}.$$

If this last relation holds, then the union on the right must contain the elements of any 2^{-n} -net of $\mathcal{M}(\hat{\mathbb{C}})$, provided that 2^{-n} is less than (half of) the Lebesgue number of the covering $\{B_{k_i}\}_{i=1}^m$. Such a net can be computed from a net of $\hat{\mathbb{C}}$ and a net of $[0, 1]$. □

4.2. Proof of Theorem A. Consider a rational map $R(z) = P(z)/Q(z)$ of degree d . The coefficients of P and Q form two $(d+1)$ -tuples of complex numbers, and we can thus specify R by a $(2d+2)$ -tuple of coefficients, or a point in \mathbb{C}^{2d+2} . It is clear that

Proposition 4.2. *$R : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a computable function if and only if there exists a computable point in \mathbb{C}^{2d+2} which specifies R .*

Let us now formulate a precise version of the Theorem A:

Theorem 4.3. *For a rational map R denote by λ_R its Brodin-Lyubich measure. Then the functional*

$$\begin{aligned} \mathcal{F} : \mathbb{C}^{2d+2} &\rightarrow \mathcal{M}(\hat{\mathbb{C}}) \\ R &\mapsto \lambda_R \end{aligned}$$

is computable.

Remark 4.4. In other words, there exists an algorithm \mathcal{A} with an oracle for $\bar{v} \in \mathbb{C}^{2d+2}$ and a single natural input n which outputs a measure $\mu \in \mathcal{D}$ which has the following property. If R is the rational map with coefficients \bar{v} then

$$W_1(\mu, \lambda_R) < 2^{-n}.$$

Of course, if R is computable, then the oracle can be replaced with an algorithm computing the coefficients of R .

Proof of the Theorem A. Let R be a rational map of degree d and ϕ be an oracle for the coefficients of R . Given $n \in \mathbb{N}$, we will show how to compute an ideal ball $B_n \subset \mathcal{M}(\hat{\mathbb{C}})$ with radius 2^{-n} containing λ_R .

Let $U \subset \mathcal{M}(\hat{\mathbb{C}})$ be the set of probability measures which are not invariant with respect to R , and let $V \subset \mathcal{M}(\hat{\mathbb{C}})$ be the set of probability measures which are not balanced. In the following, we show that, using the oracle ϕ , both U and V are lower-computable open sets.

Let us introduce a certain fixed, enumerated sequence of Lipschitz computable functions which we will use as test functions. Let \mathcal{H}_0 be the set of functions of the form:

$$\varphi_{s,r,\epsilon} = |1 - |d(x, s) - r|^+ / \epsilon|^+ \quad (4.1)$$

where s is a rational point in $\hat{\mathbb{C}}$, $r, \epsilon \in \mathbb{Q}$ and $|a|^+ = \max\{a, 0\}$. These are uniformly computable Lipschitz functions equal to 1 in the ball $B(s, r)$, to 0 outside $B(s, r + \epsilon)$ and with intermediate values in between.

Let

$$\mathcal{H} = \{\varphi_1, \varphi_2, \dots\} \quad (4.2)$$

be the smallest set of functions containing \mathcal{H}_0 and the constant 1, and closed under max, min and finite rational linear combinations. Clearly, we have:

Proposition 4.5. *\mathcal{H} is a sufficient family of uniformly computable functions.*

Moreover, the functions in \mathcal{H} are of the form $\varphi_n = c_n + g_n$ where c_n is a constant computable from n , and g_n has a bounded support, and from n one can compute a bound for its diameter.

Lemma 4.6. *The set*

$$U := \{\mu \in \mathcal{M}(\hat{\mathbb{C}}) : \mu \text{ is not invariant with respect to } R\}$$

is a lower-computable open set.

Proof. We show that U is lower-computable open by exhibiting an algorithm to semi-decide whether a probability measure μ belongs to U . By Proposition 3.11 the numbers

$$\int \varphi_i d\mu \text{ and } \int \varphi_i \circ R d\mu$$

are uniformly computable. If μ is not invariant, then there exist j such that

$$\int g_i d\mu \neq \int g_i \circ R d\mu$$

and such a j can be found. □

Lemma 4.7. *The set*

$$V := \{\mu \in \mathcal{M}(\hat{\mathbb{C}}) : \mu \text{ is not balanced with respect to } R\}$$

is a lower-computable open set.

Proof. To semi-decide whether a measure μ is not balanced, we start by enumerating all the ideal points z_i in $\hat{\mathbb{C}}$ which are not critical for R . For each z_i , we can compute an ideal ball $B_i = B(z_i, r)$ such that $R|_{B_i}$ is injective. Denote Crit_R is the set of critical values of R . Compute a rational number q such that

$$0 < q < \text{dist}(R(z_i), \text{Crit}_R).$$

The function R has a conformal inverse branch R_i^{-1} on $B(R(z_i), q)$. Compute any rational number r such that

$$0 < r < \frac{1}{4R'(z_i)}.$$

By Koebe 1/4 Theorem,

$$B_i = B(z_i, r) \subset R_i^{-1}(B(R(z_i), q)),$$

so that R is conformal on B_i .

Now, for each B_i , let $(\varphi_j^i)_j$ be the list of test functions supported on B_i . If μ is not balanced, then there exists i such that

$$\mu(R(B_i)) \neq d \cdot \mu(B_i)$$

which means that there exists l such that

$$\int \varphi_l^i \circ R_i^{-1} d\mu \neq d \cdot \int \varphi_l^i d\mu.$$

By Proposition 3.11, the numbers $\int \varphi_j^i \circ R_i^{-1} d\mu$ and $d \int \varphi_j^i d\mu$ are uniformly computable, and thus l can be found. \square

It follows that the open set $\mathcal{U} = U \cup V$ of measures which are either not invariant or not balanced is lower-computable with an oracle ϕ . Its complement is the singleton $\{\lambda_R\}$. To compute λ_R with precision 2^{-n} , enumerate all the ideal balls $B_n \subset \mathcal{M}(\hat{\mathbb{C}})$ of radius 2^{-n} and semi-decide the relation $\{\lambda_R\} \subset B_n$. This is possible because

$$\{\lambda_R\} \subset B_n \iff \mathcal{M}(\hat{\mathbb{C}}) \subset \mathcal{U} \cup B_n,$$

and the last relation is semi-decidable by Proposition 4.1. \square

4.3. A comparison of rates of convergence. In [6] two of the authors have shown:

Theorem 4.8. *There exists a computable quadratic polynomial $f_c(z) = z^2 + c$ whose Julia set J_c is not computable.*

Together with Theorem A this statement has the following amusing consequence:

Theorem 4.9 (Incommensurability of rates of convergence). *For a polynomial f_c and $z \in \mathbb{C}$ denote*

$$\phi_1(n) = \text{dist}_H((f_c)^{-n}(z), J_c) \text{ and } \phi_2(n) = W_1 \left(\frac{1}{2^n} \sum_{w \in (f_c)^{-n}(z)} \delta_w, \lambda_c \right),$$

where λ_c is the Brodin-Lyubich measure of f_c . Even though both ϕ_1 and ϕ_2 converge to 0 as $n \rightarrow \infty$, there exists a parameter c such that there **does not exist any computable function** $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $F(0) = 0$ and

$$\phi_1(n) \leq F \circ \phi_2(n).$$

Proof. If c is computable, Theorem A implies the computability of λ_c . Hence, $\phi_2(n)$ is a computable function. On the other hand, if there exists a computable bound $\phi_1(n) \xrightarrow{n \rightarrow \infty} 0$, then J_c is a computable set. Therefore, such a bound cannot exist for a parameter c as in Theorem 4.8. \square

As an illustration, consider Figure 2. The Julia set of a quadratic polynomial is rendered in gray. This particular polynomial can be written in the form $P_\theta(z) = z^2 + \exp(2\pi i \theta)z$ for $\theta = (\sqrt{5} + 1)/2$. The preimage $(P_\theta)^{-12}(z)$ (highlighted in black) for a point $z \in J(P_\theta)$ gives an excellent approximation of λ , but a very poor approximation of the whole Julia set.

4.4. Proof of Theorem B. For a given point $z \in \mathbb{C}$, set

$$\lambda_{z,m} = \frac{1}{d^m} \sum_{w \in R^{-m}(a)} \delta_w.$$

The following result is due to Drasin and Okuyama [10], and, in more generality, to Dinh and Sibony [9].

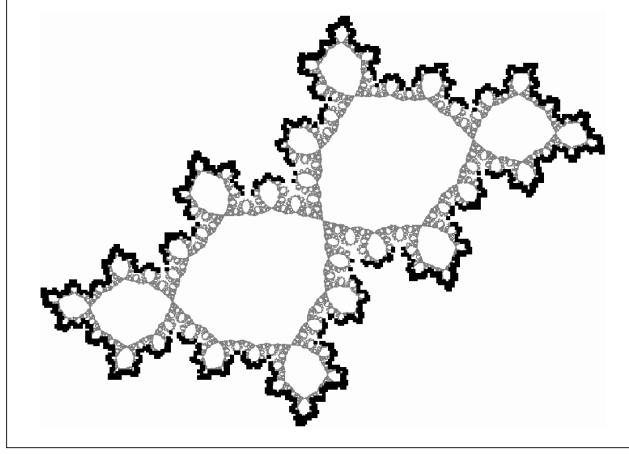


FIGURE 2. The Julia set of $P_\theta(z) = z^2 + \exp(2\pi i\theta)z$ for $\theta = (\sqrt{5} + 1)/2$. The set $(P_\theta)^{-12}(z)$ is indicated in black for a point $z \in J(P_\theta)$.

Theorem 4.10 ([10, 9]). *For each R there are constants $\alpha = \alpha(R)$ and $A = A(R)$ such that for every point $z \in \hat{\mathbb{C}}$, except at most two, and for every $f \in 1 - \text{Lip}$ the following holds:*

$$\left| \int f d\lambda_{z,m} - \int f d\lambda \right| < A\alpha^{-m}.$$

Note that A and α are independent of n, z and f .

We have then that, taking $m = Cn$ for some constant $C = C(A, \alpha)$,

$$W_1(\lambda_{z,m}, \lambda) < 2^{-n}.$$

So that in order to compute a 2^{-n} -approximation of λ_R , it is enough to compute approximations of the d^m pre-images of z by R^m . Since each pre-image can be computed in time polynomial in d^m , the entire computation can be achieved in time $O(2^{cn})$ for a $c = c(R)$.

4.5. A counter-example. In view of the above results, it is natural to ask whether a measure of maximal entropy of a computable dynamical system is *always* computable. The example below will show that this need not be the case. We will construct a map

$$T : S^1 \times S^1 \rightarrow S^1 \times S^1$$

with the following properties:

- (a) T is a computable function;
- (b) T has a measure of maximal entropy;
- (c) *every* measure of maximal entropy of T is non-computable.

We first recall a construction [11]:

Proposition 4.11 ([11]). *There exists a computable transformation $T_1 : S^1 \rightarrow S^1$ for which every invariant measure is non-computable.*

To prove this, we need the following facts:

Proposition 4.12. *Let μ be a computable Borel measure on a computable metric space X . Then the support of μ contains a computable point $x \in X$.*

Sketch of proof. We outline the proof here and leave the details to the reader. First, for each ideal ball $B = B(x, r)$ set

$$\psi_B \equiv \phi_{x, r/2, r/2}$$

as in (4.1). An exhaustive search can be used to find a sequence of ideal balls $B^i = B(x_i, r_i)$ with the following properties:

- $r_i \leq 2^{-i}$;
- $B(x_i, r_i) \subset B(x_{i-1}, 2r_{i-1})$;
- $\int \psi_{B^i} d\mu > 0$.

The algorithm can then be used to compute $x = \lim x_i \in \text{supp}(\mu)$. □

Proposition 4.13. *There exists a lower-computable open set $V \subset (0, 1)$ such that $(0, 1) \setminus V \neq \emptyset$ and V contains all computable numbers in $(0, 1)$.*

Sketch of proof. Consider an algorithm \mathcal{A} which at step m emulates the first m algorithms $\mathcal{A}_i(i)$, $i \leq m$ with respect to the Gödel ordering for m steps. That is, the i -th algorithm in the ordering is given the number i as the input parameter. From time to time, an emulated algorithm $\mathcal{A}_i(i)$ may output a rational number x_i in $(0, 1)$. Our algorithm \mathcal{A} will output an interval

$$L_i = (x_i - 3^{-i}/2, x_i + 3^{-i}/2) \cap (0, 1)$$

for each term in this sequence. The union $V = \cup L_i$ is a lower-computable set. It is easy to see from the definition of a computable real that $V \supset \mathbb{R}_C \cap (0, 1)$. If $x \in \mathbb{R}_C$ then there is a machine $\mathcal{A}_n(j)$ that on input j outputs a $3^{-j}/4$ -approximation of x . Thus the execution of $\mathcal{A}_n(n)$ will halt with an output q such that $|x - q| < 3^{-n}/4$, and x will be included in V . On the other hand, the Lebesgue measure of V is bounded by $1/2$, and thus does not cover all of $[0, 1]$. □

Sketch of proof of Proposition 4.11. By Proposition 4.13, there exists an open lower-computable set $V \subset [0, 1]$ such that the complement $K = [0, 1] \setminus V$ contains no computable points. Since V is lower-computable, there are computable sequences $\{a_i, b_i\}_{i \geq 1}$ such that $0 < a_i < b_i < 1$ and $V = \bigcup_i (a_i, b_i)$.

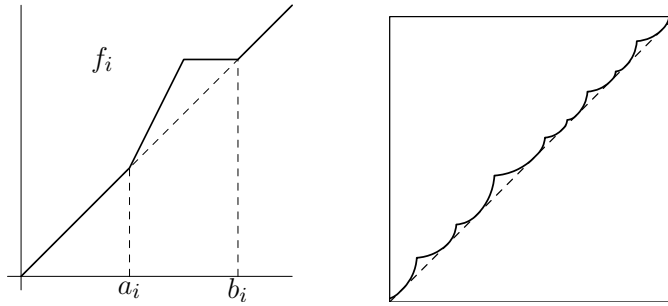


FIGURE 3. Left: a map f_i , right: the map T_1

Let us define non-decreasing, uniformly computable functions $f_i : [0, 1] \rightarrow [0, 1]$ such that

$$f_i(x) > x \text{ if } x \in (a_i, b_i) \text{ and } f_i(x) = x \text{ otherwise.}$$

For instance, we can set

$$f_i(x) = 2x - a_i \text{ on } \left[a_i, \frac{a_i + b_i}{2} \right], \text{ and}$$

$$f_i(x) = b_i \text{ on } \left[\frac{a_i + b_i}{2}, b_i \right].$$

As neither 0 nor 1 belongs to K , there is a rational number $\epsilon > 0$ such that $K \subseteq [\epsilon, 1 - \epsilon]$. Let us define $f : [0, 1] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{on } [\epsilon, 1 - \epsilon], \\ 2x - (1 - \epsilon) & \text{on } [1 - \epsilon, 1] \\ \epsilon & \text{on } [0, \epsilon] \end{cases}$$

We then define $t(x) : [0, 1] \rightarrow \mathbb{R}$ by

$$t(x) = \frac{f}{2} + \sum_{i \geq 2} 2^{-i} f_i.$$

By construction, the function $t(x)$ is computable and non-decreasing, and $t(x) > x$ if and only if $x \in [0, 1] \setminus K$. As

$$t(1) = f(1) = 1 + \epsilon = 1 + t(0),$$

we can take the quotient

$$T_1(x) \equiv t(x) \pmod{\mathbb{Z}}.$$

It is easy to see that T_1 moves all points towards the set K . More precisely, every point $x \in K$ is fixed under T_1 , and the orbit of every point $x \notin K$ converges to $\inf\{y \in K \cap [x, 1]\}$. Further, for any interval $J \in U$, all but finitely many T_1 -translates of J are disjoint from J . Hence, no finite invariant measure of T_1 can be supported on J . Thus the support of every T_1 -invariant measure is contained in K . By Proposition 4.12, no such measure can be computable. \square

We are now equipped to present the counter-example T . We define $T_2 : S^1 \rightarrow S^1$ by $T_2(x) = 2x \pmod{\mathbb{Z}}$, and set

$$T = T_1 \times T_2.$$

Firstly, by the same reasoning as above, every invariant measure μ of T is supported on $K \times S^1$ and hence is not computable by Proposition 4.12. On the other hand, T possesses invariant measures of maximal entropy. Indeed, let ν be any invariant measure of T_1 and λ the Lebesgue measure on S^1 . Setting $\mu = \nu \times \lambda$, we have

$$h_\mu(T) = h_{\text{top}}(T) = \log 2.$$

5. HARMONIC MEASURE

5.1. **Proof of Theorem C.** Let us start with several definitions.

Definition 5.1. We recall that a compact set $K \subset \hat{\mathbb{C}}$ which contains at least two points is *uniformly perfect* if the moduli of the ring domains separating K are bounded from above. Equivalently, there exists some $C > 0$ such that for any $x \in K$ and $r > 0$, we have

$$(B(x, Cr) \setminus B(x, r)) \cap K = \emptyset \implies K \subset B(x, r).$$

In particular, every connected set is uniformly perfect.

It is known that:

Theorem 5.1 (see [17]). *The Julia set of a polynomial P of degree $d \geq 2$ is a uniformly perfect compact set.*

Recall that the *logarithmic capacity* $\text{Cap}(\cdot)$ has been defined in Section 2.3. We next define:

Definition 5.2. Let $\Omega \subset \hat{\mathbb{C}}$ be an open and connected domain and set $J = \partial\Omega$. We say that Ω satisfies the *capacity density condition* if there exists a constant $C > 0$ such that

$$\text{Cap}(B(x, r) \cap \partial\Omega) \geq Cr \quad \text{for all } x \in \partial\Omega \text{ and } r \leq r_0. \quad (5.1)$$

We note:

Theorem 5.2 (see Theorem 1 in [22]). *Condition (5.1) is equivalent to uniform perfectness of $\partial\Omega$.*

The celebrated result of Kakutani [14], gives a connection between Brownian motion and the Harmonic measure.

Theorem 5.3. [14] *Let $K \subset \hat{\mathbb{C}}$ be a compact set with a connected complement Ω . Fix a point $x \in \Omega$ and let B_t denote a Brownian path started at x . Let the random variable T be the first moment when B_t hits $\partial\Omega$, and let $\omega_x = \omega_{\Omega, x}$ denote the harmonic measure corresponding to x . Then for any measurable function f on $\partial\Omega$,*

$$\int f d\omega_x = \mathbb{E}[f(B_T)].$$

In [2] the following computable version of the Dirichlet problem has been proved:

Theorem 5.4. *Let $K \subset \hat{\mathbb{C}}$ be a compact computable set with a connected complement Ω . Let $x \in \Omega$ be any point in Ω and let B_t denote a Brownian path started at x . There is an algorithm \mathcal{A} that, given access to K , x , and a precision parameter ε , outputs an $\varepsilon/4$ -approximate sample β_ε from a random variable B_{T_ε} where T_ε is a stopping rule on B_t that always satisfies*

$$\varepsilon/2 < \text{dist}(B_{T_\varepsilon}, K) < \varepsilon.$$

In other words, we are able to stop the Brownian motion at distance $\approx \varepsilon$ from the boundary. We now formulate the following proposition that is a reformulation of Theorem C:

Proposition 5.5. *Let $\Omega \subset \hat{\mathbb{C}}$ be the complement of a computable compact set K and x_0 be a point in Ω . Suppose Ω is connected and satisfies the capacity density condition. Then the harmonic measure ω_{Ω, x_0} is computable with an oracle for x_0 .*

Proof. Fix $x_0 \in \Omega$. As before, we denote by (B_t) the Brownian Motion started at x_0 and set

$$T = \min\{t : B_t \in \partial\Omega\}.$$

We will use Theorem 5.4 together with the capacity density condition to prove Proposition 5.5.

The capacity density condition implies the following (see [12], page 343):

Proposition 5.6. *There exists a constant $\nu = \nu(C)$ (with C as in the capacity density condition) such that for any $\eta > 0$ the following holds. Let $y \in \Omega$ be a point such that $\text{dist}(y, \omega) \leq \eta$, and let B^ω be a Brownian Motion started at y . Let*

$$T^y := \min\{t : B_t^y \in \partial\Omega\}$$

be the first time B^y hits the boundary of Ω . Then

$$\mathbb{P}[|B_{T_y}^y - y| \geq 2\eta] < \nu. \quad (5.2)$$

In other words, there is at least a constant probability that the first point where B^y hits the boundary is close to the starting point y .

Now let f be any function on K satisfying the 1-Lip condition. Our goal is to compute

$$\int f d\omega = \mathbb{E}[f(B_T)].$$

within any prescribed precision parameter δ . Note that

$$\mathbb{E}[f(B_T)] = \mathbb{E}_{B_{T_\varepsilon}}[\mathbb{E}[f(B_T)|B_{T_\varepsilon}]].$$

We first claim that we can compute an ε such that

$$|\tilde{f}_{2\varepsilon}(B_{T_\varepsilon}) - \mathbb{E}[f(B_T)|B_{T_\varepsilon}]| < \delta/2. \quad (5.3)$$

Here T_ε is given by the any stopping rule as in Proposition 5.4, and $\tilde{f}_{2\varepsilon}(B_{T_\varepsilon})$ is *any* evaluation of f in a 2ε -neighborhood of B_{T_ε} (note that f itself is not defined on $B_{T_\varepsilon} \notin K$).

Let M be a universal bound on the absolute value of f . By (5.2) we can compute an $\varepsilon < \delta/20$ such that if y is ε -close to K , the probability that $|B_{T_y}^y - y| > \delta/10$ is smaller than $\delta/10M$. We split the probabilities into two cases: one where B_T stays $\delta/10$ -close to B_{T_ε} , and the complementary case. By (5.3) we have

$$\begin{aligned} |\tilde{f}_{2\varepsilon}(B_{T_\varepsilon}) - \mathbb{E}[f(B_T)|B_{T_\varepsilon}]| &= \\ |\tilde{f}_{2\varepsilon}(B_{T_\varepsilon}) - \mathbb{E}[f(B_T)|B_{T_\varepsilon}], |B_T - B_{T_\varepsilon}| < \delta/10| &\cdot \mathbb{P}[|B_T - B_{T_\varepsilon}| < \delta/10] + \\ |\tilde{f}_{2\varepsilon}(B_{T_\varepsilon}) - \mathbb{E}[f(B_T)|B_{T_\varepsilon}], |B_T - B_{T_\varepsilon}| \geq \delta/10| &\cdot \mathbb{P}[|B_T - B_{T_\varepsilon}| \geq \delta/10] \leq \\ &(\delta/6) \cdot 1 + M \cdot (\delta/10M) < \delta/2. \end{aligned}$$

To complete the proof of the proposition it remains to note that given a β_ε that $\varepsilon/4$ -approximates B_{T_ε} as in Theorem 5.4, we can evaluate $\tilde{f}_{2\varepsilon}(B_{T_\varepsilon})$ by evaluating $\tilde{f}_{3\varepsilon/2}(\beta_\varepsilon)$ (by evaluating f at any point in a $3\varepsilon/2$ -neighborhood of β_ε). In this way, we obtain

$$\mathbb{E}_{\beta_\varepsilon}[\tilde{f}_{3\varepsilon/2}(\beta_\varepsilon) - \mathbb{E}[f(B_T)]] < \delta/2. \quad (5.4)$$

Thus, being able to evaluate $\tilde{f}_{3\varepsilon/2}(\beta_\varepsilon)$ with precision $\delta/2$ suffices. \square

5.2. A counter-example. As demonstrated by the following example, even for a computable regular domain, the harmonic measure is not necessarily computable. Thus the capacity density condition in Theorem C is crucial.

For $a, b \in \mathbb{R}$, we denote by $[a, b]$ the shortest arc of the unit circle between $e^{2\pi ia}$ and $e^{2\pi ib}$. As before, let \mathcal{A}_i be the Gödel ordering of algorithms. Define a collection of subsets of the circle as follows. If \mathcal{A}_n halts in l steps, we set $j = \max(l, 8n)$ and denote by

$$L_n := L_n^j := \bigcup_{k=1-2^j+2^{j-8n}}^{2^j-2^{j-8n}-1} [2^{-n} + k \cdot 2^{-2n-j}, 2^{-n} + k \cdot 2^{-2n-j} + \exp(-2^{2n+2j})].$$

Otherwise, if \mathcal{A}_n does not halt, we denote

$$L_n := L_n^\infty := [2^{-n} - 2^{-2n} + 2^{-10n}, 2^{-n} + 2^{-2n} - 2^{-10n}]$$

(see Figure 4).

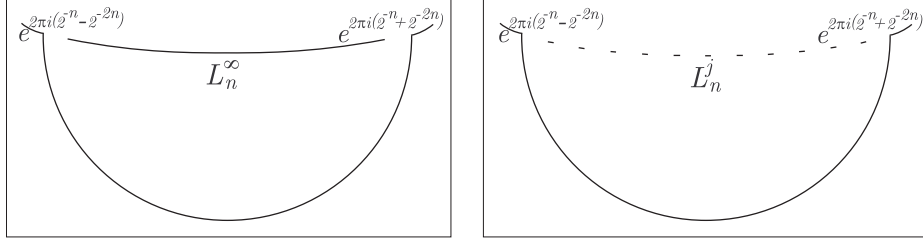


FIGURE 4. Left: a set L_n^∞ corresponding to an algorithm \mathcal{A}_n which does not halt; right: a set L_n^j corresponding to \mathcal{A}_n which halts in j steps.

Let D_n denote the disk whose diameter is given by the points $\exp(2\pi i(2^{-n} - 2^{-2n}))$ and $\exp(2\pi i(2^{-n} + 2^{-2n}))$. Let

$$\Lambda := \mathbb{D} \cup \bigcup_{n=10}^{\infty} D_n.$$

The domain Ω is obtained by removing the arcs from L_n from Λ . To be precise, set

$$\Omega := \Lambda \setminus \bigcup_{n=10}^{\infty} L_n.$$

and

$$K := \mathbb{C} \setminus \Omega.$$

We note:

Proposition 5.7. *The compact set K is computable.*

Proof. Note that

$$\text{dist}_H(L_n^j, L_n^\infty) < 12 \cdot 2^{-2n-j}.$$

We can thus compute L_n with an arbitrary precision by emulating \mathcal{A}_n for sufficiently many steps.

To compute the set K with precision 2^{-m} , it suffices to compute the first 2^{m+4} sets L_n with precision $2^{-(m+4)}$. \square

Now let us show that:

Proposition 5.8. *The harmonic measure*

$$\omega := \omega_{\Omega,0}$$

is not computable.

For a set $K_0 \subset \{z : 0 < \delta < |z| < r < 1\}$ set

$$\gamma(K_0) := -\log \text{Cap}(K_0).$$

We need an auxilliary lemma:

Lemma 5.9 (Theorem 5.1.4 in [23]). *If K_1, \dots, K_n are compact subsets of the unit disk, then*

$$\frac{1}{\gamma(K_1 \cup \dots \cup K_n)} \leq \frac{1}{\gamma(K_1)} + \dots + \frac{1}{\gamma(K_n)}.$$

Let S_n be the part of the boundary of the disk D_n lying outside of \mathbb{D} , $S_n := \partial D_n \setminus \mathbb{D}$. Harmonic measure is always non-atomic, so if ω is computable, then $\omega(S_n)$ is also computable. We show:

Proposition 5.10. *If \mathcal{A}_n does not halt, then $\omega(S_n) < 2^{-9n+2}$. If \mathcal{A}_n halts, $\omega(S_n) > 2^{-2n-3}$.*

Proof. As before, let B_t be the Brownian motion started at 0 and let T denote the hitting time of $\partial\Omega$,

$$T := \inf\{t : B_t \in \partial\Omega\}.$$

Let us recall that for $E \subset \partial\Omega$ we have

$$\omega(E) = \mathbb{P}[B_T \in E].$$

Assume now that \mathcal{A}_n does not halt. Let us introduce a new domain $\Omega' := \mathbb{C} \setminus [2^{-n} - 2^{-2n}, 2^{-n} + 2^{-2n}]$ and T' be the corresponding hitting time. Observe that if $B_T \in S_n$ then

$$B_{T'} \in K_n := [2^{-n} - 2^{-2n}, 2^{-n} - 2^{-2n} + 2^{-10n}] \cup [2^{-n} + 2^{-2n} - 2^{-10n}, 2^{-n} + 2^{-2n}].$$

Thus

$$\omega_{\Omega,0}(S_n) = \mathbb{P}[B_T \in S_n] \leq \mathbb{P}[B_{T'} \in K_n] = \omega_{\Omega',0}(K_n).$$

The desired estimate is now obtained by mapping $(\Omega', 0)$ conformally to $(\mathbb{D}, 0)$.

Assume that \mathcal{A}_n halts in j steps. To bound $\omega(S_n)$ in this case, we will use the following estimate on harmonic measure ([12], Equation (III.9.2)):

Let $K_0 \subset \{z : 0 < \delta < |z| < r < 1\}$. Then

$$\omega_{\mathbb{D} \setminus K_0,0}(K_0) \leq \frac{\log\left(\frac{1}{\delta}\right)}{\gamma(K_0) + \log(1 - r^2)} \quad (5.5)$$

Let T'' denote the hitting time of $\partial\mathbb{D}$ by B_t , and let

$$M_n := \{z \in L_n^\infty : \text{dist}(z, \partial\Omega) > 2^{-2n-j-4}\}$$

be the part of the arc L_n^∞ lying relatively far away from the boundary.

Conformally mapping D_n to the unit disk centered at $z \in M_n$ and using the estimate (5.5) and Lemma 5.9, we obtain that for $z \in M_n$ we have

$$\omega_{D_n \setminus L_n, z}(L_n) \leq 2^{-j+3} < 1/8 \quad (5.6)$$

We will also need $T_1 \geq T''$ – the first hitting time of ∂D_n after hitting ∂D , and $T_2 \leq T_1$ – the first hitting time of $\partial D_n \cup L_n$ after hitting ∂D .

Note now that

$$\mathbb{P}[B_{T''} \in M_n] = \text{length}(M_n)/2\pi \geq 2^{-2n-1} \quad (5.7)$$

Let us note that by symmetry and estimate (5.6), we have

$$\begin{aligned} \mathbb{P}[B_T \in S_n \mid B_{T''} \in M_n] &\geq \mathbb{P}[B_{T_1} \in S_n \mid B_{T''} \in M_n] - \mathbb{P}[B_{T_2} \in L_n \mid B_{T''} \in M_n] = \\ &\frac{1}{2} - \mathbb{P}[B_{T_2} \in L_n \mid B_{T''} \in M_n] \geq \frac{1}{2} - \max_{z \in M_n} \omega_{D_n \setminus L_n, z}(L_n) > 1/4. \end{aligned} \quad (5.8)$$

The desired lower estimate on ω is now obtained by combining (5.7) and (5.8). \square

We now conclude the proof of Proposition 5.8:

Proof of Proposition 5.8. Assume the contrary, that is, suppose that ω is computable. For every $n \in \mathbb{N}$ let $\{U_j^n(z)\}_{j=1}^\infty$ be a sequence of functions given by:

$$U_j^n(z) = \begin{cases} 1 & \text{if } \text{dist}(z, S_n) < 2^{-j}; \\ 0 & \text{if } \text{dist}(z, S_n) > 2 \cdot 2^{-j}; \\ 1 - 2^j(d - 2^{-j}) & \text{if } d = \text{dist}(z, S_n) \in [2^{-j}, 2 \cdot 2^{-j}] \end{cases}$$

We have:

- (a) the functions $U_j^n(z)$ are computable uniformly in n and j .

Since ω is non-atomic,

- (b) for a fixed n we have

$$\int U_j^n(z) d\omega > \omega(S_n) \text{ and } \int U_j^n(z) d\omega \xrightarrow{j \rightarrow \infty} \omega(S_n).$$

Similarly, we can construct a sequence of functions $L_j^n(z)$ such that

- (c) the functions $L_j^n(z)$ are computable uniformly in n and j ;

- (d) for a fixed n we have

$$\int L_j^n(z) d\omega < \omega(S_n) \text{ and } \int L_j^n(z) d\omega \xrightarrow{j \rightarrow \infty} \omega(S_n).$$

We leave the details of the second construction to the reader.

By part (3) of Proposition 3.10, properties (a) and (c) imply that the integrals

$$\int U_j^n(z) d\omega \text{ and } \int L_j^n(z) d\omega$$

are uniformly computable. Consider an algorithm $\mathcal{A}_{\text{halt}}$ which upon inputting a natural number n does the following:

- (1) $j := 1$;
- (2) evaluate u_j, l_j such that

$$|u_j - \int U_j^n(z) d\omega| < 2^{-20n} \text{ and } |l_j - \int L_j^n(z) d\omega| < 2^{-20n};$$

- (3) if $u_j < 2^{-9n+2} + 2^{-19n}$ then output 0 and halt;
- (4) if $l_j > 2^{-2n-3} - 2^{-19n}$ then output 1 and halt;
- (5) $j := j + 1$ and go to (2).

By Proposition 5.10 and properties (b) and (d), we have the following:

- if \mathcal{A}_n halts then $\mathcal{A}_{\text{halt}}$ outputs 1 and halts, and
- if \mathcal{A}_n does not halt then $\mathcal{A}_{\text{halt}}$ outputs 0 and halts.

Thus $\mathcal{A}_{\text{halt}}$ is an algorithm solving the Halting Problem, which contradicts the algorithmic unsolvability of the Halting Problem. \square

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